

Wiretap channel capacity: Secrecy criteria, strong converse, and phase change

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Abstract—This paper employs equal-image-size source partitioning techniques to derive the capacities of the general discrete memoryless wiretap channel (DM-WTC) under four different secrecy criteria. These criteria respectively specify requirements on the expected values and tail probabilities of the differences, in absolute value and in exponent, between the joint probability of the secret message and the eavesdropper's observation and the corresponding probability if they were independent. Some of these criteria reduce back to the standard leakage and variation distance constraints that have been previously considered in the literature. The capacities under these secrecy criteria are found to be different when non-vanishing error and secrecy tolerances are allowed. Based on these new results, we are able to conclude that the strong converse property generally holds for the DM-WTC only under the two secrecy criteria based on constraining the tail probabilities. Under the secrecy criteria based on the expected values, an interesting phase change phenomenon is observed as the tolerance values vary.

I. INTRODUCTION

The discrete memoryless wiretap channel (DM-WTC) $(\mathcal{X}, P_{Y,Z|X}, \mathcal{Y} \times \mathcal{Z})$ consists of a sender X , a legitimate receiver Y , and an eavesdropper Z . A message M is to be sent reliably from X to Y and discreetly against eavesdropping by Z . Over n uses of the DM-WTC, let $f^n : \mathcal{M} \rightarrow \mathcal{X}^n$ and $\varphi^n : \mathcal{Y}^n \rightarrow \mathcal{M}$ be the encoding and decoding functions respectively employed at X and Y , where $\mathcal{M} = [1 : 2^{nR}]$ is the message set and M is uniformly distributed over \mathcal{M} . The transmission reliability requirement is specified by

$$\Pr \{\varphi^n(Y^n) \neq M\} \leq \epsilon_n \quad (1)$$

where $\epsilon_n \in (0, 1)$ denotes the error tolerance. The secrecy requirement assesses how much one may learn about M from Z^n . This requirement is often quantified by measuring the level of “independence” between M and Z^n based on either the variation distance

$$\begin{aligned} & \frac{1}{2} \|P_{M,Z^n} - P_M P_{Z^n}\|_1 \\ & \triangleq \frac{1}{2} \sum_{(m,z^n) \in \mathcal{M} \times \mathcal{Z}^n} |P_{M,Z^n}(m, z^n) - P_M(m)P_{Z^n}(z^n)| \end{aligned}$$

or the divergence $D(P_{M,Z^n} \| P_M P_{Z^n}) = I(M; Z^n)$ between P_{M,Z^n} and $P_M P_{Z^n}$. Another way of quantifying the secrecy

requirement is to view the problem as a binary hypothesis testing of the alternate hypothesis of M and Z^n being independent against the null hypothesis of M and Z^n being correlated. This is an interesting case in which we would like the false positive probability given by the likelihood ratio test

$$\begin{aligned} & P_{M,Z^n} \left(\left\{ (m, z^n) \in \mathcal{M} \times \mathcal{Z}^n : \frac{P_M(m)P_{Z^n}(z^n)}{P_{M,Z^n}(m, z^n)} \geq \tau \right\} \right) \\ & \rightarrow 1 \quad \text{as } n \rightarrow \infty \end{aligned}$$

where the decision threshold $\tau \in [0, 1)$ serves as a measure of secrecy with $\tau \rightarrow 1$ begin the most secret situation. Note that the log-likelihood $\log_2 \frac{P_M(m)P_{Z^n}(z^n)}{P_{M,Z^n}(m, z^n)}$ may also be used in the hypothesis testing problem above.

For every $(m, z^n) \in \mathcal{M} \times \mathcal{Z}^n$, define

$$v(m, z^n) \triangleq \begin{cases} \left| 1 - \frac{P_M(m)P_{Z^n}(z^n)}{P_{M,Z^n}(m, z^n)} \right|^+ & \text{if } P_{M,Z^n}(m, z^n) > 0 \\ 0 & \text{if } P_{M,Z^n}(m, z^n) = 0 \end{cases}$$

where $|c|^+$ equals c if $c > 0$ and 0 otherwise, and

$$i(m, z^n) \triangleq \begin{cases} -\log_2 \frac{P_M(m)P_{Z^n}(z^n)}{P_{M,Z^n}(m, z^n)} & \text{if } P_{M,Z^n}(m, z^n) > 0 \\ 0 & \text{if } P_{M,Z^n}(m, z^n) = 0. \end{cases}$$

We see that all the secrecy requirements discussed above can be compactly specified in terms of the tail probabilities and expected values of $v(M, Z^n)$ and $i(M, Z^n)$:

$$\begin{aligned} \mathbf{S}_1(\delta_n) &: P_{M,Z^n}(\{v(M, Z^n) > \delta_n\}) \rightarrow 0 \\ \mathbf{S}_2(\delta_n) &: E_{M,Z^n}[v(M, Z^n)] = \|P_{M,Z^n} - P_M P_{Z^n}\|_1 \leq \delta_n \\ \mathbf{S}_3(l_n) &: P_{M,Z^n}(\{i(M, Z^n) > l_n\}) \rightarrow 0 \\ \mathbf{S}_4(l_n) &: E_{M,Z^n}[i(M, Z^n)] = I(M; Z^n) \leq l_n \end{aligned}$$

where $\delta_n \in (0, 1]$, $l_n \in (0, \infty)$, and $E_{M,Z^n}[\cdot]$ denotes the expectation w.r.t. P_{M,Z^n} . Note that $\mathbf{S}_2(\delta_n)$ and $\mathbf{S}_4(l_n)$ are the variation distance and divergence (leakage) constraints, respectively, while $\mathbf{S}_1(\delta_n)$ and $\mathbf{S}_3(l_n)$ correspond to the secrecy requirements specified by the hypothesis testing problem using the likelihood and log-likelihood ratios, respectively.

Clearly these secrecy requirements are related to each other. For example, $\mathbf{S}_1(\delta_n)$ implies $\mathbf{S}_2(\delta_n)$, and $\mathbf{S}_3(l_n)$ implies $\mathbf{S}_4(l_n)$. Then, by Pinsker's inequality, $\mathbf{S}_4(l_n)$ implies $\mathbf{S}_2\left(\sqrt{\frac{l_n \ln 2}{2}}\right)$ if $l_n \in (0, \frac{2}{\ln 2})$. Similarly by Markov's inequality, $\mathbf{S}_2(\delta_n)$ implies $\mathbf{S}_1(\sqrt{\delta_n})$ if $\delta_n \rightarrow 0$. Also by elementary logarithm inequalities, $\mathbf{S}_1(\delta_n)$ implies $\mathbf{S}_3(\frac{2\delta_n}{\ln 2})$ if $\delta_n \rightarrow 0$. Thus for vanishing tolerances, \mathbf{S}_1 – \mathbf{S}_4 are essentially equivalent.

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Special cases of these secrecy requirements have been considered in the literature. For example, requiring $\epsilon_n \rightarrow 0$ in (1), $\mathbf{S}_4(nr_l)$ is the equivocation constraint originally considered in [1]. Six secrecy requirements \mathbf{S}_1 – \mathbf{S}_6 are more recently considered in [2]. Setting $\epsilon_n \rightarrow 0$, \mathbf{S}_1 is $\mathbf{S}_4(l_n)$ for some $l_n \rightarrow 0$, \mathbf{S}_2 is $\mathbf{S}_2(\delta_n)$ for some $\delta_n \rightarrow 0$, \mathbf{S}_3 is $\mathbf{S}_3(l_n)$ for some $l_n \rightarrow 0$, \mathbf{S}_4 is $\mathbf{S}_4(l_n)$ for some $\frac{l_n}{n} \rightarrow 0$, and \mathbf{S}_6 is $\mathbf{S}_3(l_n)$ for some $\frac{l_n}{n} \rightarrow 0$.

The majority of known secrecy capacity results under the above secrecy requirements are for cases with vanishing error tolerance, $\epsilon_n \rightarrow 0$, and secrecy tolerance, $l_n \rightarrow 0$, $\frac{l_n}{n} \rightarrow 0$, or $\delta_n \rightarrow 0$. These results are nicely summarized in [2], which shows that the secrecy capacities under \mathbf{S}_1 – \mathbf{S}_6 (see footnote 1) of the DM-WTC are all given by $\max_{P_{U,X}} I(U; Y) - I(U; Z)$,

where $U \rightarrow X \rightarrow Y, Z$. Here we are mainly interested in cases where both the error tolerance ϵ_n and secrecy tolerance δ_n , l_n or $\frac{l_n}{n}$ are non-vanishing, on which only a few partial results exist. The oldest such result dates back to Wyner's original paper [1], in which the secrecy capacity under $\mathbf{S}_4(nr_l)$ of the degraded DM-WTC ($P_{Y,Z|X} = P_{Z|Y}P_{Y|X}$) is calculated for the case of $\epsilon_n \rightarrow 0$. The ϵ -secrecy capacity under $\mathbf{S}_4(l_n)$ of the degraded DM-WTC is obtained in [3] for the case of $\frac{l_n}{n} \rightarrow 0$. This case has also been extended to the general DM-WTC in [4] and [5]. The ϵ -secrecy capacity under $\mathbf{S}_2(\delta)$ of the degraded DM-WTC is found in [6].

In this paper, we determine the secrecy capacities for the general DM-WTC under the above four security requirements, \mathbf{S}_1 – \mathbf{S}_4 , with non-vanishing tolerances. All of these results can be straightforwardly obtained using our recently developed equal-image-size source partitioning techniques [4], [7]. Further, the ϵ -secrecy capacity for each of these four requirements is unique. Under \mathbf{S}_1 and \mathbf{S}_3 the *strong converse* property holds, while it does not under \mathbf{S}_2 and \mathbf{S}_4 in general. In addition, under \mathbf{S}_2 and \mathbf{S}_4 , the capacity can be broken into distinct phases depending on the error tolerance. For instance, under \mathbf{S}_2 the capacity of the channel is either equal to the capacity of the channel with vanishing error, or the capacity of the channel with no secrecy requirement. We call this interesting phenomenon a *phase change*.

II. MAIN RESULTS

For $i \in \{1, 2, 3, 4\}$, we call (f^n, φ^n) a $(n, R_n, \epsilon_n, \mathbf{S}_i(\eta_n))$ -code if the domain of f^n (i.e., \mathcal{M}) is of cardinality 2^{nR_n} , and the pair satisfy both (1) and $\mathbf{S}_i(\eta_n)$. Further we say the *rate error secrecy* (RES)-triple $(a, b, c) \in \mathbb{R}^3$ is \mathbf{S}_i -achievable if there exists a sequence of $(n, R_n, \epsilon_n, \mathbf{S}_i(\eta_n))$ -codes such that $\lim_{n \rightarrow \infty} (R_n, \epsilon_n, \eta_n) = (a, b, c)$ if $i \in \{1, 2\}$, and $\lim_{n \rightarrow \infty} (R_n, \epsilon_n, \frac{\eta_n}{n}) = (a, b, c)$ if $i \in \{3, 4\}$. Then the ϵ -secrecy capacity under the appropriate $\mathbf{S}_i(\cdot)$ is the maximum R such that the RES-triple (R, ϵ, η) is \mathbf{S}_i -achievable.

Note that for \mathbf{S}_3 and \mathbf{S}_4 , the above definition corresponds to what is called “weak” secrecy in the literature [2]. If “strong” secrecy is desired, the definition could be modified to that the RES-triple (R, ϵ, η) is \mathbf{S}_i -achievable when there exists a sequence of $(n, R_n, \epsilon_n, \mathbf{S}_i(\eta_n))$ -codes such

that $\lim_{n \rightarrow \infty} (R_n, \epsilon_n, \eta_n) = (R, \epsilon, \eta)$, for $i \in \{3, 4\}$. We have instead chosen to present the “weak” versions of these criteria, simply because their proofs trivially recover their “strong” counterparts. That is the ϵ -secrecy capacity under $\mathbf{S}_3(l)$ or $\mathbf{S}_4(l)$, must be less than the ϵ -secrecy capacity under $\mathbf{S}_3\left(n\frac{\log_2 n}{n}\right)$ and $\mathbf{S}_4\left(n\frac{\log_2 n}{n}\right)$ -respectively. Yet, the secrecy capacities under $\mathbf{S}_3\left(n\frac{\log_2 n}{n}\right)$ and $\mathbf{S}_4\left(n\frac{\log_2 n}{n}\right)$ can also be achieved under $\mathbf{S}_3(l_n)$ and $\mathbf{S}_4(l_n)$ for some $l_n \rightarrow 0$, using [8, Theorem 17.11].

Write

$$C_1 \triangleq \max_{P_X} I(X; Y)$$

$$C_2 \triangleq \max_{P_{U,X}} I(U; Y) - I(U; Z)$$

where the alphabet \mathcal{U} of U in the definition of C_2 can be assumed to have $|\mathcal{U}| \leq |\mathcal{X}|$. Also restrict $\epsilon \in [0, 1)$ and $\delta \in [0, 1]$, and $r_l \in [0, \infty)$. Then the following theorems give our main results regarding the secrecy capacities:

Theorem 1. *The ϵ -secrecy capacity under $\mathbf{S}_1(\delta)$ of the DM-WTC is given by*

$$\mathbb{C}_1(\delta) \triangleq \begin{cases} C_2 & \text{if } \delta < 1 \\ C_1 & \text{otherwise.} \end{cases}$$

for all ϵ .

Theorem 2. *The ϵ -secrecy capacity under $\mathbf{S}_2(\delta)$ of the DM-WTC is given by*

$$\mathbb{C}_2(\epsilon, \delta) \triangleq \begin{cases} C_2 & \text{if } \epsilon + \delta < 1 \\ C_1 & \text{otherwise.} \end{cases}$$

Theorem 3. *The ϵ -secrecy capacity under $\mathbf{S}_3(nr_l)$ of the DM-WTC is given by*

$$\mathbb{C}_3(r_l) \triangleq C_2 + \min(r_l, C_1 - C_2)$$

for all ϵ .

Theorem 4. *The ϵ -secrecy capacity under $\mathbf{S}_4(nr_l)$ of the DM-WTC is given by*

$$\mathbb{C}_4(\epsilon, r_l) \triangleq C_2 + \min\left(\frac{r_l}{1 - \epsilon}, C_1 - C_2\right).$$

Theorems 1 and 4 state that the ϵ -secrecy capacities of the DM-WTC under $\mathbf{S}_1(\delta)$ and $\mathbf{S}_3(nr_l)$ are invariant to the value of $\epsilon \in [0, 1)$ for all valid values of δ and r_l , respectively. In other words, the strong converse property holds under $\mathbf{S}_1(\delta)$ and $\mathbf{S}_3(nr_l)$. Although invariant of the error tolerance, the ϵ -secrecy capacity under $\mathbf{S}_3(nr_l)$ is non-trivially dependent on the leakage rate r_l . In specific, the ϵ -secrecy capacity under $\mathbf{S}_3(nr_l)$ increases linearly as a function of r_l from C_2 until it saturates at C_1 , the (non-secret) capacity of the discrete memoryless channel (DMC) $(\mathcal{X}, P_{Y|X}, \mathcal{Y})$.

When the secrecy requirements are weakened to $\mathbf{S}_2(\delta)$ and $\mathbf{S}_4(nr_l)$ from $\mathbf{S}_1(\delta)$ and $\mathbf{S}_3(nr_l)$ respectively, Theorems 2 and 4 show that the strong converse property no longer holds for the DM-WTC as the ϵ -secrecy capacities generally depend

on the value of ϵ . Under $\mathbf{S}_2(\delta)$, the ϵ -secrecy capacity remains at C_2 as long as $\epsilon \in [0, 1 - \delta)$. However, for $\epsilon \in [1 - \delta, 1)$, the ϵ -secrecy capacity value experiences an abrupt phase change, increasing to C_1 as if there is no secrecy requirement. Restricting to within either of the two value ranges, the ϵ -secrecy capacity under $\mathbf{S}_2(\delta)$ is invariant to ϵ .

Under $\mathbf{S}_4(nr_l)$, the ϵ -secrecy capacity remains at C_2 when $r_l = 0$ for all $\epsilon \in [0, 1)$. Note that this also includes the cases of strong secrecy ($\mathbf{S}_4(l_n)$ with $l_n \rightarrow 0$) and bounded leakage ($\mathbf{S}_4(l_n)$ with $l_n = l$). Thus the strong converse property holds when $r_l = 0$ as proven in [4] and [5]. For any fixed $r_l \in (0, C_1 - C_2)$, the ϵ -secrecy capacity increases from $C_2 + r_l$ to C_1 and then levels off as ϵ increases in the range $[0, 1)$. The DM-WTC exhibits a phase change from where the strong converse property holds to where it does not. When $r_l \geq C_1 - C_2$, the ϵ -secrecy capacity value remains at C_1 for all $\epsilon \in [0, 1)$, and the DM-WTC exhibits another phase change after which the strong converse property holds again.

III. PROOFS OF THEOREMS

We prove the converses in Theorems 1–4 by employing the following strong Fano's inequality developed in [4] and information stabilization result developed in [7]:

Strong Fano's inequality. *For any (f^n, φ^n) of rate R that gives $\Pr\{\varphi^n(Y^n) \neq M\} \leq \epsilon$ over the DM-WTC, there exist a random index Q_n that ranges over an index set \mathcal{Q} whose cardinality is at most polynomial in n , $\zeta_n \rightarrow 0$, and an index subset*

$$\mathcal{Q}_n^R \triangleq \left\{ q_n \in \mathcal{Q}_n : R \leq \frac{1}{n} I(M; Y^n | Q_n = q_n) + \zeta_n \right\}$$

satisfying $P_{Q_n}(\mathcal{Q}_n^R) \geq 1 - \epsilon - \zeta_n$.

Information stabilization. *For the (f^n, φ^n) pair, random index Q_n , and index set \mathcal{Q}_n above, there exist $\xi_n \rightarrow 0$ and an index subset $\mathcal{Q}_n^Z \subseteq \mathcal{Q}_n$ satisfying $P_{Q_n}(\mathcal{Q}_n^Z) \geq 1 - \xi_n$:¹*

- 1) $P_{Z^n|Q_n}(\hat{\mathcal{Z}}^n(q_n)|q_n) \geq 1 - \xi_n$, where $\hat{\mathcal{Z}}^n(q_n) \triangleq \{z^n \in \mathcal{Z}^n : P_{Z^n|Q_n}(z^n|q_n) \geq \xi_n 2^{-H(Z^n|Q_n=q_n)}\}$,
- 2) there exists a $\hat{\mathcal{M}}(q_n) \subseteq \mathcal{M}$ satisfying $P_{M|Q_n}(\hat{\mathcal{M}}(q_n)|q_n) \geq 1 - \xi_n$, and $P_{M|Q_n}(m|q_n) \geq \xi_n 2^{-H(M|Q_n=q_n)}$ for each $m \in \hat{\mathcal{M}}(q_n)$, and
- 3) $P_{Z^n|M, Q_n}(\hat{\mathcal{Z}}^n(m, q_n)|m, q_n) \geq 1 - \xi_n$ where $\hat{\mathcal{Z}}^n(m, q_n) \triangleq \{z^n \in \mathcal{Z}^n : P_{Z^n|M, Q_n}(z^n|m, q_n) \geq \xi_n 2^{-H(Z^n|M, Q_n=q_n)}\}$,

for each $q_n \in \mathcal{Q}_n^Z$.

Obtained through the information stabilization result in the appendix, the following lemma will also be needed:

Lemma 5. *For any $r \geq 0$, there exist $\tau_n \rightarrow 0$, $\mu_n \rightarrow 0$, and $\lambda_n \rightarrow 0$ satisfying $n\lambda_n \rightarrow \infty$ such that by defining*

$$\mathcal{Q}_n^S(r) \triangleq \left\{ q_n \in \mathcal{Q}_n : \frac{1}{n} I(M; Z^n | Q_n = q_n) \leq r + \tau_n \right\}$$

¹For any non-negative $\lambda_n \rightarrow 0$, $a_n > 0$, and $b_n > 0$, $a_n \doteq_{\lambda_n} b_n$ means $|\frac{1}{n} \log_2 a_n - \frac{1}{n} \log_2 b_n| \leq \lambda_n$.

and

$$\Omega_n(r) \triangleq \left\{ (m, z^n) \in \mathcal{M} \times \mathcal{Z}^n : \right.$$

$$\left. P_{M, Z^n}(m, z^n) \leq 2^{n(r + \lambda_n)} P_M(m) P_{Z^n}(z^n) \right\},$$

then

$$P_{M, Z^n}(\Omega_n(r)) \leq P_{Q_n}(\mathcal{Q}_n^S(r)) + \mu_n.$$

For proving achievability in Theorems 2 and 4, we will make use of the following lemma to simplify discussions:

Lemma 6. *For $i \in \{2, 4\}$, if the RES-triple $(R, 0, \eta)$ is \mathbf{S}_i -achievable, then the RES-triple $(R, \gamma, (1 - \gamma)\eta)$ is also \mathbf{S}_i -achievable for any $\gamma \in [0, 1)$.*

A. Proof of Theorem 1

(Direct) For any $\delta \in [0, 1)$ and $\epsilon \in [0, 1)$, the RES-triple (C_2, ϵ, δ) being \mathbf{S}_1 -achievable follows directly from [8, Theorem 17.11], which in particular shows the RES-triple $(C_2, 0, 0)$ is \mathbf{S}_1 -achievable. On the other hand, the RES-triple $(C_1, \epsilon, 1)$ is \mathbf{S}_1 -achievable since C_1 is the channel capacity for the DMC $(\mathcal{X}, P_{Y|X}, \mathcal{Y})$, and $\delta = 1$ corresponds to no secrecy constraint.

(Converse) To prove that $\mathbf{C}_1(\delta)$ is an upper bound on the ϵ -secrecy capacity under $\mathbf{S}_1(\delta)$, first apply Lemma 5 to obtain values τ_n , μ_n , and λ_n which converge to 0 as n increases, such that $P_{M, Z^n}(\Omega_n(0)) \leq P_{Q_n}(\mathcal{Q}_n^S(0)) + \mu_n$, for sets $\Omega_n(0)$ and $\mathcal{Q}_n^S(0)$ as defined in Lemma 5. We also have that $P_{M, Z^n}(\Omega_n(0)) \geq 1 - \rho_n$ for some $\rho_n \rightarrow 0$, due to $\mathbf{S}_1(\delta)$. Thus $\mathbf{S}_1(\delta)$ and Lemma 5 together imply that

$$P_{Q_n}(\mathcal{Q}_n^S(0)) \geq P_{M, Z^n}(\Omega_n(0)) - \mu_n \geq 1 - \rho_n - \mu_n. \quad (2)$$

But then the strong Fano's inequality and (2) together give the existence of a $q_n \in \mathcal{Q}_n$ such that

$$R \leq \frac{1}{n} I(M; Y^n | Q_n = q_n) + \zeta_n \quad (3)$$

$$\frac{1}{n} I(M; Z^n | Q_n = q_n) \leq \tau_n \quad (4)$$

since $P_{Q_n}(\mathcal{Q}_n^R \cap \mathcal{Q}_n^S(0)) \geq 1 - \epsilon - \zeta_n - \rho_n - \mu_n > 0$ for large enough n and $\epsilon \in [0, 1)$. Combining (3) and (4) with [8, Lemma 17.12], it follows that $R \leq C_2 + \zeta_n + \eta_n$ for all $\epsilon \in [0, 1)$. On the other hand, when $\delta = 1$, the strong Fano's inequality (i.e., (3)) gives $R \leq C_1 + \zeta_n$ for all ϵ , as in the standard strong converse argument for the DMC $(\mathcal{X}, P_{Y|X}, \mathcal{Y})$.

B. Proof of Theorem 2

(Direct) The RES-triple (C_2, ϵ, δ) is \mathbf{S}_2 -achievable, once again, by [8, Theorem 17.11], for $\epsilon + \delta < 1$. For $\epsilon + \delta \geq 1$, the RES-triple $(C_1, \epsilon, 1 - \epsilon)$ is \mathbf{S}_2 -achievable by Lemma 6, since the RES-triple $(C_1, 0, 1)$ is \mathbf{S}_2 -achievable.

(Converse) On the other hand, to prove that $\mathbf{C}_2(\epsilon, \delta)$ is an upper bound on the ϵ -secrecy capacity under $\mathbf{S}_2(\delta)$, observe that $\mathbf{S}_2(\delta)$ implies

$$\delta \geq \|P_{M, Z^n} - P_M P_{Z^n}\|_1$$

$$\begin{aligned}
&\geq \sum_{(m, z^n) \in \mathcal{M} \times \mathcal{Z}^n \setminus \Omega_n(0)} P_{M, Z^n}(m, z^n) - P_M(m) P_{Z^n}(z^n) \\
&\geq \sum_{(m, z^n) \in \mathcal{M} \times \mathcal{Z}^n \setminus \Omega_n(0)} P_{M, Z^n}(m, z^n) (1 - 2^{-n\lambda_n}) \\
&= [1 - 2^{-n\lambda_n}] [1 - P_{M, Z^n}(\Omega_n(0))].
\end{aligned} \tag{5}$$

Thus combining Lemma 5 and (5) gives

$$P_{Q_n}(\mathcal{Q}_n^S(0)) \geq 1 - \delta - 2^{-n\lambda_n} - \mu_n.$$

As a result, if $\epsilon + \delta < 1$, then there must exist a $q_n \in \mathcal{Q}_n$ such that (3) and (4) are simultaneously satisfied since

$$P_{Q_n}(\mathcal{Q}_n^R \cap \mathcal{Q}_n^S(0)) \geq 1 - \epsilon - \delta - \zeta_n - 2^{-n\lambda_n} - \mu_n > 0$$

for all sufficiently large n . Hence again by [8, Lemma 17.12], we have $R \leq C_2 + \zeta_n + \tau_n$. If $\epsilon + \delta \geq 1$, then the strong Fano's inequality (i.e., (3)) gives $R \leq C_1 + \zeta_n$.

C. Proof of Theorem 3

(Direct) The RES-triple $((C_2 + \min(r_l, C_1 - C_2), \epsilon, r_l)$ is \mathbf{S}_3 achievable due to [8, Theorem 17.13], which shows that the RES-triple $((C_2 + \min(r_l, C_1 - C_2), 0, r_l)$ is \mathbf{S}_3 achievable.

(Converse) On the other hand, to prove that $\mathbb{C}_3(r_l)$ is an upper bound on the ϵ -secrecy capacity under $\mathbf{S}_3(nr_l)$ of the DM-WTC, we note that Lemma 5 and $\mathbf{S}_3(nr_l)$ directly imply

$$P_{Q_n}(\mathcal{Q}_n^S(r_l)) \geq P_{M, Z^n}(\Omega_n(r_l)) - \mu_n \geq 1 - \rho_n - \mu_n \quad (6)$$

for some $\rho_n \rightarrow 0$. Thus as before the strong Fano's inequality and (6) together give the existence of a $q_n \in \mathcal{Q}_n$ satisfying (3) and

$$\frac{1}{n} I(M; Z^n | Q_n = q_n) \leq r_l + \tau_n \quad (7)$$

since $P_{Q_n}(\mathcal{Q}_n^R \cap \mathcal{Q}_n^S(r_l)) \geq 1 - \epsilon - \zeta_n - \rho_n - \mu_n > 0$. Hence by applying [8, Lemma 17.12] to the combination of (3) and (7), we have $R < C_2 + r_l + \zeta_n + \tau_n$. The strong Fano's inequality also gives the bound $R \leq C_1 + \zeta_n$. Thus we obtain $R \leq \min\{C_2 + r_l + \zeta_n + \tau_n, C_1 + \zeta_n\}$ for all ϵ .

D. Proof of Theorem 4

(Direct) First note the RES-triple $((C_2 + \min(\frac{r_l}{1-\epsilon}, C_1 - C_2), 0, \frac{r_l}{1-\epsilon})$ is \mathbf{S}_4 achievable due to [8, Theorem 17.13]. Hence the RES-triple $((C_2 + \min(\frac{r_l}{1-\epsilon}, C_1 - C_2), \epsilon, r_l)$ is \mathbf{S}_4 -achievable by Lemma 6.

(Converse) To prove $\mathbb{C}_4(\epsilon, r_l)$ upper-bounds the ϵ -secrecy capacity under $\mathbf{S}_4(nr_l)$ of the DM-WTC, notice that $\mathbf{S}_4(nr_l)$ implies

$$\begin{aligned}
r_l &\geq \frac{1}{n} I(M; Z^n) \geq \frac{1}{n} I(M; Z^n | Q_n) - \frac{\alpha}{n} \log_2 n \\
&\geq \sum_{q_n \in \mathcal{Q}_n^R} \frac{1}{n} I(M; Z^n | Q_n = q_n) P_{Q_n}(q_n) - \frac{\alpha}{n} \log_2 n \\
&\geq \min_{q_n \in \mathcal{Q}_n^R} \frac{1}{n} I(M; Z^n | Q_n = q_n) P_{Q_n}(\mathcal{Q}_n^R) - \frac{\alpha}{n} \log_2 n \quad (8)
\end{aligned}$$

where n^α is the cardinality bound on \mathcal{Q}_n . But from the strong Fano's inequality, we have $P_{Q_n}(\mathcal{Q}_n^R) \geq 1 - \epsilon - \zeta_n$. This

together with (8) implies that there must be a $q_n \in \mathcal{Q}_n^R$ such that

$$\frac{1}{n} I(M; Z^n | Q_n = q_n) \leq \frac{r_l + \frac{\alpha}{n} \log_2 n}{1 - \epsilon - \zeta_n}. \quad (9)$$

Again by the strong Fano's inequality, for this q_n we also have (3). Hence combining (3) and (8) with [8, Lemma 17.12], it follows that

$$R \leq C_2 + \frac{r_l + \frac{\alpha}{n} \log_2 n}{1 - \epsilon - \zeta_n} + \zeta_n.$$

The strong Fano's inequality alone also gives the bound $R \leq C_1 + \zeta_n$. Thus we have

$$R \leq \min \left\{ C_2 + \frac{r_l + \frac{\alpha}{n} \log_2 n}{1 - \epsilon - \zeta_n} + \zeta_n, C_1 + \zeta_n \right\}.$$

IV. CONCLUSIONS

Employing the recently developed techniques of equal-image-size partitioning, we obtained the ϵ -secrecy capacities under $\mathbf{S}_1(\delta)$, $\mathbf{S}_2(\delta)$, $\mathbf{S}_3(nr_l)$, and $\mathbf{S}_4(nr_l)$ of the DM-WTC for non-vanishing ϵ , δ , and r_l . The secrecy criteria considered include the standard leakage and variation distance secrecy constraints often employed in the literature. Our new results show that both the capacity value and the strong converse property of the DM-WTC are in fact dependent on the secrecy criterion adopted. We conjecture that the interesting phase change phenomenon observed in cases where the strong converse property does not hold is commonplace in many other multi-terminal DMCs.

APPENDIX

A. Proof of Lemma 5

We need the following lemma to prove Lemma 5:

Lemma 7. Let Q_n be a random index ranging over \mathcal{Q}_n , whose cardinality is at most polynomial in n , and V be any discrete random variable distributed over \mathcal{V} . Then there exist $\lambda_n \rightarrow 0$ and $\xi'_n \rightarrow 0$ such that $n\lambda_n \rightarrow \infty$ and

$$\begin{aligned}
P_{V, Q_n}(\{(v, q_n) \in \mathcal{V} \times \mathcal{Q}_n : P_{V|Q_n}(v|q_n) \doteq_{\lambda_n} P_V(v)\}) \\
\geq 1 - \xi'_n.
\end{aligned}$$

Note that λ_n and ξ'_n both depend only on the polynomial cardinality bound on \mathcal{Q}_n .

Proof: Let $\alpha > 0$ be such that $|\mathcal{Q}_n| \leq n^\alpha$. First write $\mathcal{A} = \{(v, q_n) \in \mathcal{V} \times \mathcal{Q}_n : P_{V|Q_n}(v|q_n) > n^{2\alpha} P_V(v)\}$ and $\mathcal{B} = \{(v, q_n) \in \mathcal{V} \times \mathcal{Q}_n : P_{V|Q_n}(v|q_n) < n^{-2\alpha} P_V(v)\}$. Then

$$\begin{aligned}
P_{V, Q_n}(\{(v, q_n) \in \mathcal{V} \times \mathcal{Q}_n : P_{V|Q_n}(v|q_n) \doteq_{\lambda_n} P_V(v)\}) \\
\geq 1 - P_{V, Q_n}(\mathcal{A} \cup \mathcal{B})
\end{aligned} \tag{10}$$

where $\lambda_n = \frac{2\alpha}{n} \log_2 n$. Thus the lemma is verified by (10) if we can show that $P_{V, Q_n}(\mathcal{A} \cup \mathcal{B}) \rightarrow 0$. In particular, we do so by bounding $P_{V, Q_n}(\mathcal{A}) \leq n^{-\alpha}$ and $P_{V, Q_n}(\mathcal{B}) \leq n^{-2\alpha}$, and setting $\xi'_n = n^{-\alpha} + n^{-2\alpha}$.

To bound $P_{V, Q_n}(\mathcal{A})$, note that for all $(v, q_n) \in \mathcal{A}$,

$$P_{Q_n}(q_n) \leq n^{-2\alpha} \tag{11}$$

since

$$P_V(v) \geq P_{V|Q_n}(v|q_n)P_{Q_n}(q_n) \geq n^{2\alpha}P_V(v)P_{Q_n}(q_n).$$

Then the upper bound on $P_{V,Q_n}(\mathcal{A})$ follows from (11) as below:

$$\begin{aligned} P_{V,Q_n}(\mathcal{A}) &= \sum_{(v,q_n) \in \mathcal{A}} P_{V|Q_n}(v|q_n)P_{Q_n}(q_n) \\ &\leq \sum_{(v,q_n) \in \mathcal{A}} P_{V|Q_n}(v|q_n)n^{-2\alpha} \leq n^{-\alpha}. \end{aligned}$$

The upper bound on $P_{V,Q_n}(\mathcal{B})$ follows similarly in that

$$\begin{aligned} P_{V,Q_n}(\mathcal{B}) &= \sum_{(v,q_n) \in \mathcal{B}} P_{V|Q_n}(v|q_n)P_{Q_n}(q_n) \\ &\leq \sum_{(v,q_n) \in \mathcal{B}} P_V(v)P_{Q_n}(q_n)n^{-2\alpha} \leq n^{-2\alpha}. \end{aligned}$$

Apply Lemma 7 three times with $V = M$, $V = Z^n$, and $V = (M, Z^n)$, respectively. Writing

$$\begin{aligned} \Gamma_n &\triangleq \left\{ (m, z^n, q_n) \in \mathcal{M} \times \mathcal{Z}^n \times \mathcal{Q}_n : \right. \\ &\quad P_{M,Z^n|Q_n}(m, z^n|q_n) \doteq_{\lambda_n} P_{M,Z^n}(m, z^n), \\ &\quad P_{M|Q_n}(m|q_n) \doteq_{\lambda_n} P_M(m), \text{ and} \\ &\quad \left. P_{Z^n|Q_n}(z^n|q_n) \doteq_{\lambda_n} P_{Z^n}(z^n) \right\} \end{aligned}$$

where λ_n is obtained in Lemma 7, we have

$$P_{M,Z^n,Q_n}(\Gamma_n) \geq 1 - 3\xi'_n. \quad (12)$$

Next define

$$\begin{aligned} \Xi_n &\triangleq \left\{ (m, z^n, q_n) \in \mathcal{M} \times \mathcal{Z}^n \times \mathcal{Q}_n : q_n \in \mathcal{Q}_n^Z, \right. \\ &\quad \left. m \in \tilde{\mathcal{M}}(q_n), \text{ and } z^n \in \hat{\mathcal{Z}}^n(q_n) \cap \tilde{\mathcal{Z}}^n(m, q_n) \right\} \end{aligned}$$

with the corresponding \mathcal{Q}_n^Z , $\tilde{\mathcal{M}}(q_n)$, $\hat{\mathcal{Z}}^n(q_n)$, and $\tilde{\mathcal{Z}}^n(m, q_n)$ as given in the information stabilization result summarized in Section III. Similar to before,

$$P_{M,Z^n,Q_n}(\Xi_n) \geq 1 - 4\xi_n. \quad (13)$$

Combining (12) and (13) gives

$$P_{M,Z^n,Q_n}(\Xi_n \cap \Gamma_n) \geq 1 - 3\xi'_n - 4\xi_n. \quad (14)$$

From here note that for any $(m, z^n, q_n) \in \Xi_n \cap \Gamma_n$,

$$P_{M,Z^n}(m, z^n) \leq 2^{n(r+\lambda_n)}P_M(m)P_{Z^n}(z^n)$$

implies

$$\frac{1}{n} \log_2 \frac{P_{Z^n,M|Q_n}(z^n, m|q_n)}{P_{Z^n|Q_n}(z^n|q_n)P_{M|Q_n}(m|q_n)} \leq r + 4\lambda_n, \quad (15)$$

because $(m, z^n, q_n) \in \Gamma_n$. And then in turn, for all $(m, z^n, q_n) \in \Gamma_n \cap \Xi_n$,

$$r + 4\lambda_n \geq \frac{1}{n} I(M; Z^n | Q_n = q_n) - 2\xi_n \quad (16)$$

since $(m, z^n, q_n) \in \Xi_n$. Thus Lemma 5 results from (16) by setting $\tau_n = 4\lambda_n + 2\xi_n$ and $\mu_n = 3\xi'_n + 4\xi_n$, because we have from (14)

$$\begin{aligned} P_{M,Z^n}(\Omega_n(r)) &\leq P_{M,Z^n,Q_n}(\Xi_n \cap \Gamma_n \cap \Omega_n(r) \times \mathcal{Q}_n) + 3\xi'_n + 4\xi_n \\ &\leq P_{Q_n}(\mathcal{Q}_n^S(r)) + 3\xi'_n + 4\xi_n. \end{aligned}$$

B. Proof of Lemma 6

For $i \in \{2, 4\}$, we can construct a $(n, R_n, (1-\gamma)\epsilon_n + \gamma, \mathbf{S}_i((1-\gamma)l_n))$ -code (\hat{f}^n, φ^n) , given that there exists a $(n, R_n, \epsilon_n, \mathbf{S}_i(l_n))$ -code (f^n, φ^n) . Whence the lemma follows by the definition of the RES-triples. Letting \hat{M} be a random variable distributed identical, but independent, to M . The new encoder, \hat{f}^n , is constructed by setting it equal to $f(M)$ with probability $1-\gamma$ and to $f(\hat{M})$ with probability γ . While the new decoder $\hat{\varphi}^n = \varphi^n$.

Clearly, an error will likely occur if $\hat{f}(\hat{M})$ is set equal to $f(\hat{M})$. On the other hand, the probability of error will revert to that of (f^n, φ^n) if $\hat{f}(\hat{M})$ is set equal to $f(M)$. Thus the probability of error for $(\hat{f}^n, \hat{\varphi}^n)$ is at most $(1-\gamma)\epsilon_n + \gamma$.

Letting $P_{Z^n,M}$ be the joint distribution of Z^n, M for induced by f^n , we can write the joint distribution of Z^n, M for \hat{f}^n as $(1-\gamma)P_{Z^n,M} + \gamma P_{Z^n}P_M$, while the marginals remain P_M and P_{Z^n} . But then, for the variation distance,

$$\begin{aligned} &\|(1-\gamma)P_{Z^n,M} + \gamma P_{Z^n}P_M - P_{Z^n}P_M\|_1 \\ &= (1-\gamma)\|P_{Z^n,M} - P_{Z^n}P_M\|_1. \end{aligned}$$

And for divergence

$$\begin{aligned} D((1-\gamma)P_{Z^n,M} + \gamma P_{Z^n}P_M \| P_{Z^n}P_M) &\leq (1-\gamma)D(P_{Z^n,M} \| P_{Z^n}P_M) + \gamma D(P_{Z^n}P_M \| P_{Z^n}P_M) \\ &= (1-\gamma)D(P_{Z^n,M} \| P_{Z^n}P_M). \end{aligned}$$

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